C THE BINOMIAL EXPANSION

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Consider $(a+b)^n$ where n is a positive integer.

$\boldsymbol{a+b}$ is called a **binomial** as it contains two terms.

The **binomial expansion** of $(a+b)^n$ is obtained by writing the expression without brackets.

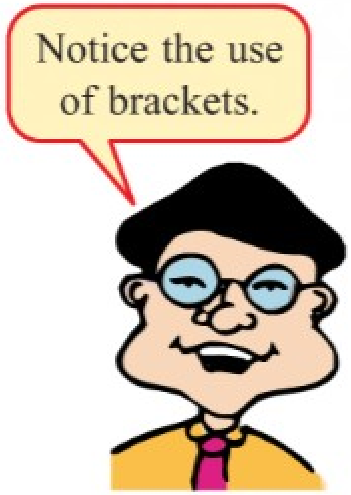
$$\boldsymbol{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}.$$

$$\begin{aligned}
(a+b)^3 &= (a+b)^2(a+b) \\
&= (a^2+2ab+b^2)(a+b) \\
&= a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3 \\
&= a^3 + 3a^2b + 3ab^2 + b^3
\end{aligned}$$

Example 7 **Self Tutor**

Expand and simplify using the rule $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$:

a $(x+4)^3$ **b** $(3x-2)^3$


Notice the use of brackets.

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Expand and simplify using the rule $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$:

a $(x+4)^3$

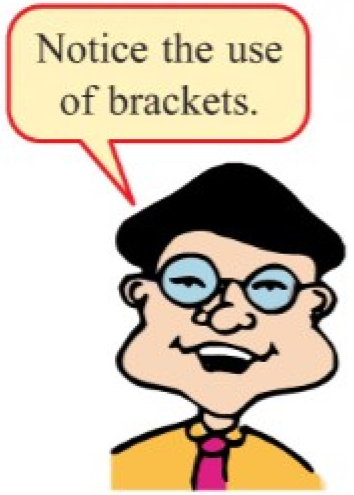
b $(3x-2)^3$

a We substitute $a = x$ and $b = 4$.

$$\begin{aligned} \therefore\ (x+4)^3 &= x^3 + 3 \times x^2 \times 4 + 3 \times x \times 4^2 + 4^3 \\ &= x^3 + 12x^2 + 48x + 64 \end{aligned}$$

b We substitute $a = (3x)$ and $b = (-2)$.

$$\begin{aligned} \therefore\ (3x-2)^3 &= (3x)^3 + 3 \times (3x)^2 \times (-2) + 3 \times (3x) \times (-2)^2 + (-2)^3 \\ &= 27x^3 - 54x^2 + 36x - 8 \end{aligned}$$


Notice the use of brackets.

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

a $(x+1)^3$
b $(x+3)^3$
c $(x+5)^3$
d $(x+y)^3$
e $(x-1)^3$
f $(x-5)^3$
g $(x-4)^3$
h $(x-y)^3$
i $(2+y)^3$
j $(2x+1)^3$
k $(3x+1)^3$
l $(2y+3x)^3$
m $(2-y)^3$
n $(2x-1)^3$
o $(3x-1)^3$
p $(2y-3x)^3$

2 By expanding and simplifying $(a+b)^3(a+b)$, show that $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

3 Use the binomial expansion $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

a $(x+y)^4$
b $(x+1)^4$
c $(x+2)^4$
d $(x+3)^4$
e $(x-y)^4$
f $(x-1)^4$
g $(x-2)^4$
h $(2x-1)^4$

4 Consider:

$$\begin{aligned} (a+b)^1 &= a + b \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ (a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \end{aligned}$$

The expressions on the right hand side of each identity contain the coefficients:

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

This triangle of numbers is called **Pascal's triangle**.

a Predict the next two rows of Pascal's triangle, and explain how you found them.

b Hence, write down the binomial expansion for:

i $(a+b)^5$ **ii** $(a-b)^5$ **iii** $(a+b)^6$ **iv** $(a-b)^6$

c **i** Expand and simplify $(x-2)^5$.

ii Check your answer by substituting $x = 1$ into your expansion.

1 **a** x^3+3x^2+3x+1 **b** $x^3+9x^2+27x+27$

c $x^3+15x^2+75x+125$ **d** $x^3+3x^2y+3xy^2+y^3$

e x^3-3x^2+3x-1 **f** $x^3-15x^2+75x-125$

g $x^3-12x^2+48x-64$ **h** $x^3-3x^2y+3xy^2-y^3$

i $8+12y+6y^2+y^3$ **j** $8x^3+12x^2+6x+1$

k $27x^3+27x^2+9x+1$

l $8y^3+36xy^2+54x^2y+27x^3$

m $8-12y+6y^2-y^3$ **n** $8x^3-12x^2+6x-1$

o $27x^3-27x^2+9x-1$

p $8y^3-36xy^2+54x^2y-27x^3$

3 **a** $x^4+4x^3y+6x^2y^2+4xy^3+y^4$

b $x^4+4x^3+6x^2+4x+1$

c $x^4+8x^3+24x^2+32x+16$

d $x^4+12x^3+54x^2+108x+81$

e $x^4-4x^3y+6x^2y^2-4xy^3+y^4$

f $x^4-4x^3+6x^2-4x+1$

g $x^4-8x^3+24x^2-32x+16$

h $16x^4-32x^3+24x^2-8x+1$

4 **a**

1 5 10 10 5 1
1 6 15 20 15 6 1

We start and finish each row with a 1. The other entries are obtained by adding the two adjacent numbers in the row above.

b **i** $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

ii $a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$

iii $a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$

iv $a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$

c **i** $x^5-10x^4+40x^3-80x^2+80x-32$

ii When $x=1$, $(x-2)^5=(-1)^5=-1$

and $$\begin{aligned}&1^5-10\times1^4+40\times1^3-80\times1^2+80\times1-32\\&=1-10+40-80+80-32\\&=-1 \quad ✓\end{aligned}$$